The Bethe Ansatz

Heisenberg Model and Generalizations

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Introduction

Bethe ansatz

- Hans Bethe (1931): particular parametrization of eigenstates of the 1D Heisenberg model Bethe, ZS. f. Phys. (1931)
- Today: generalized to whole class of 1D quantum many-body systems

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- Today: generalized to whole class of 1D quantum many-body systems

Although eigenvalues and eigenstates of a finite system may be obtained from brute force numerical diagonalization

Two important advantages of the Bethe ansatz

- all eigenstates characterized by set of quantum numbers $\rightarrow$ distinction according to specific physical properties
- in many cases: possibility to take thermodynamic limit, no system size restrictions

One shortcoming

- structure of obtained eigenstates in practice often to complicated to obtain correlation functions

Contents

Ferromagnetic 1D Heisenberg model

Goal

obtain exact eigenvalues and eigenstates with their physical properties

$$
\begin{aligned}
H &= -J \sum_{n=1}^{N} \mathbf{S}_n \cdot \mathbf{S}_{n+1} \\
&= -J \sum_{n=1}^{N} \left[\frac{1}{2} \left(S_n^+ S_{n+1}^- + S_n^- S_{n+1}^+ \right) + S_n^z S_{n+1}^z \right]
\end{aligned}
$$

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Basic remarks: eigenstates

- reference basis: $\{|\sigma_1 \ldots \sigma_N\rangle\}$
- Bethe ansatz is basis tansformation
- rotational symmetry z-axis $S_T^z \equiv \sum_{n=1}^N S_n^z$ conserved: $[H, S_T^z] = 0$
  $\Rightarrow$ block diagonalization by sorting basis according to $\langle S_T^z \rangle = N/2 - r$
  where r = number of down spins

Intuitive states

Lowest energy states intuitively obtained

- block $r = 0$: **groundstate**

$$|F\rangle \equiv |\uparrow \ldots \uparrow\rangle$$

  with energy $E_0 = -JN/4$

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- block $r = 1$: **one-particle excitations**

$$|\psi\rangle = |k\rangle \equiv \sum_{n=1}^{N} a(n)|n\rangle \quad \text{where} \quad a(n) \equiv \frac{1}{\sqrt{N}} e^{ikn} \quad \text{and} \quad |n\rangle \equiv S_n^- |F\rangle$$

  with energy $E = J(1 - \cos k) + E_0$

magnons $|k\rangle$

N one-particle excitations correspond to elementary particles “magnons” with one particle states $|k\rangle$

Note: not the lowest excitations!

Systematic proceeding to obtain eigenstates

- block $r = 1$: dim$= N$

$$|\psi\rangle = \sum_{n=1}^{N} a(n)|n\rangle$$

$H|\psi\rangle = E|\psi\rangle \Leftrightarrow$

$$2[E - E_0]a(n) = J[2a(n) - a(n-1) - a(n+1)]$$

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- block $r = 2$: dim$= \binom{N}{2} = N(N-1)/2$

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$H|\psi\rangle = E|\psi\rangle \Leftrightarrow$

$$\begin{aligned} 2[E - E_0]a(n_1, n_2) = J[4a(n_1, n_2) - a(n_1-1, n_2) \\ - a(n_1+1, n_2) - a(n_1, n_2-1) - a(n_1, n_2+1)] \quad \text{for} \quad n_2 > n_1+1 \end{aligned}$$

$$2[E - E_0]a(n_1, n_2) = J[2a(n_1, n_2) - a(n_1-1, n_2) - a(n_1, n_2+1)] \quad \text{for} \quad n_2 = n_1+1$$

Two magnon excitations – eigenstates

Solution by parametrization

$$a(n_1, n_2) = Ae^{i(k_1 n_1 + k_2 n_2)} + A' e^{i(k_1 n_2 + k_2 n_1)}$$

where

$$\frac{A}{A'} \equiv e^{i\theta} = -\frac{e^{i(k_1+k_2)} + 1 - 2e^{ik_1}}{e^{i(k_1+k_2)} + 1 - 2e^{ik_2}}$$

with energy $E = J(1 - \cos k_1) + J(1 - \cos k_2) + E_0$

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To summarize rewrite:

$$a(n_1, n_2) = e^{i(k_1 n_1 + k_2 n_2 + \frac{1}{2}\theta)} + e^{i(k_1 n_2 + k_2 n_1 - \frac{1}{2}\theta)} \quad \text{where} \quad 2\cot\frac{\theta}{2} = \cot\frac{k_1}{2} - \cot\frac{k_2}{2}$$

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Translational invariance:

$$Nk_1 = 2\pi\lambda_1 + \theta, \qquad Nk_2 = 2\pi\lambda_2 - \theta \quad \text{where} \quad \lambda_i \in \{0, 1, \ldots, N-1\}$$

with λ_i the integer (Bethe) quantum numbers

Two magnon excitations – eigenstates

Rewrite constraints

$$2\cot\frac{\theta}{2} = \cot\frac{k_1}{2} - \cot\frac{k_2}{2}$$
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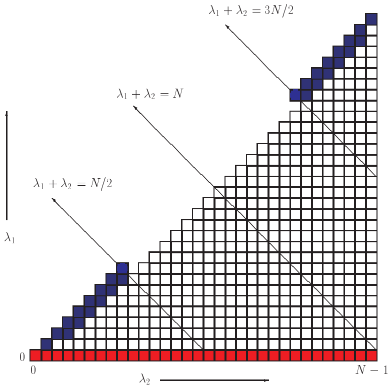
$$Nk_2 = 2\pi\lambda_2 - \theta$$

$N(N-1)/2$ solutions:

- class 1 (red): $\lambda_1 = 0$
  $\Rightarrow k_1 = 0, k_2 = 2\pi\lambda_2/N, \theta = 0$
- class 2 (white): $\lambda_2 - \lambda_1 \geq 2$
  $\Rightarrow$ real solutions k_1, k_2
- class 3 (blue): $\lambda_2 - \lambda_1 < 2$
  $\Rightarrow$ complex solutions
  $k_1 \equiv \frac{k}{2} + iv, k_2 \equiv \frac{k}{2} - iv$

Figure for $N = 32$ Karbach and Müller, Computers in Physics (1997)



Two magnon excitations – dispersion

$$Nk_1 = 2\pi\lambda_1 + \theta \qquad Nk_2 = 2\pi\lambda_2 - \theta$$
$$\Rightarrow k = k_1 + k_2 = 2\pi(\lambda_1 + \lambda_2)/N$$

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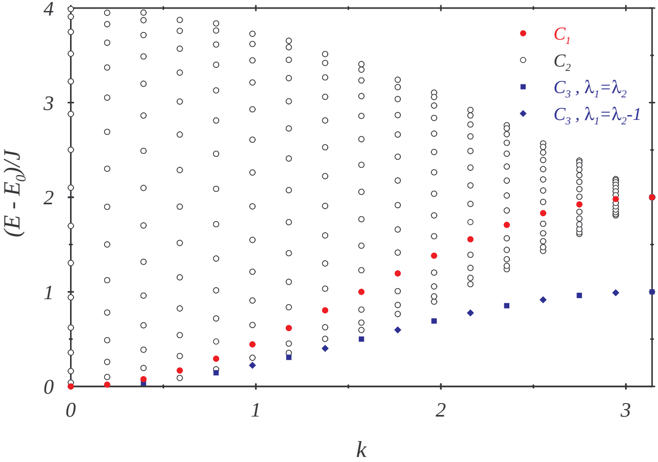

Two magnon excitations – physical properties

classification

- class 1 + 2: almost free scattering states, i.e. for $N \to \infty$ degenerate with direct product of two non-interacting magnons
- class 3: bound states

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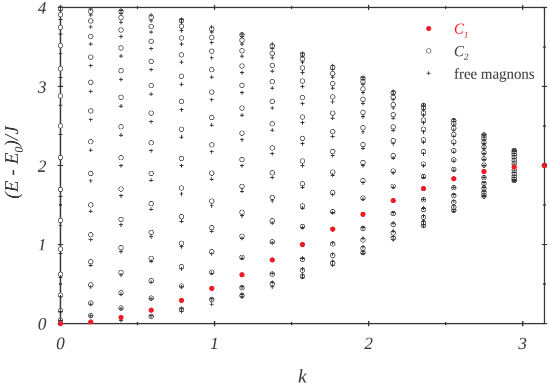

Two magnon excitations – class 3: bound states

dispersion in thermodynamic limit ($N \to \infty$): $E = \frac{J}{2}(1 - \cos k) + E_0$

Two magnon excitations – class 3: bound states

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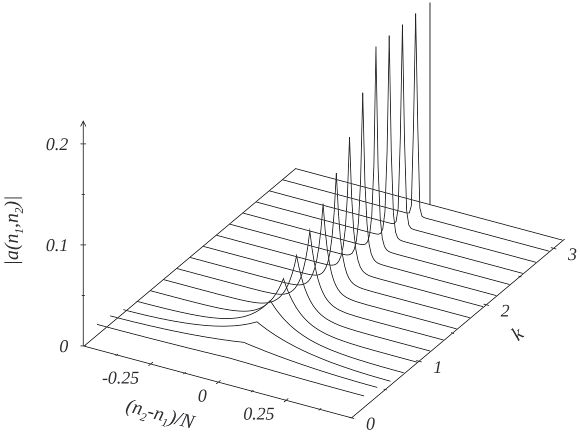
Figure for $N = 128$ Karbach and Müller, Computers in Physics (1997)



Eigenstates – states with $r > 2$

$|\psi\rangle = \sum_{1 \leq n_1 < \ldots < n_r \leq N} a(n_1, \ldots, n_r) |n_1, \ldots, n_r\rangle$

where $\quad a(n_1, \ldots, n_r) = \sum_{\mathcal{P} \in S_r} \exp\left(i \sum_{j=1}^{r} k_{\mathcal{P}j} n_j + \frac{i}{2} \sum_{i<j} \theta_{\mathcal{P}i\mathcal{P}j} \right)$

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energy: $\quad E = J \sum_{j=1}^{r} (1 - \cos k_j) + E_0$

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quantum numbers: $\lambda_i \in \{0, 1, \ldots, N-1\}$ determined via

$Nk_i = 2\pi\lambda_i + \sum_{j \neq i} \theta_{ij}$ and $2 \cot \frac{\theta_{ij}}{2} = \cot \frac{k_i}{2} - \cot \frac{k_j}{2}$ for $i, j = 1, \ldots, r$

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Solution becomes tedious for $N, r \gg 1$, but

to analyze specific physical properties, it is sufficient to study particular solutions

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to analyze specific physical properties, it is sufficient to study particular solutions

Bound states

bound states (class 3) in all subspaces r with dispersion $E = \frac{J}{r}(1 - \cos k) + E_0$
→ lowest energy excitations
→ pure many-body feature

Contents

Antiferromagnetic 1D Heisenberg model

$$
\begin{aligned}
H &= J \sum_{n=1}^{N} \mathbf{S}_n \cdot \mathbf{S}_{n+1} \\
&= J \sum_{n=1}^{N} \left[\frac{1}{2} \left(S_n^+ S_{n+1}^- + S_n^- S_{n+1}^+ \right) + S_n^z S_{n+1}^z \right]
\end{aligned}
$$

Spectrum

Eigenvalues inversed as compared to ferromagnetic Heisenberg model, e.g. $|F\rangle \equiv |\uparrow \ldots \uparrow\rangle$ state with highest energy

Goals

- ground-state $|A\rangle$
- magnetic field
- excitations

Ground-state

Classical candidate (no eigenstate): **Néel state**

$$|\mathcal{N}_1\rangle \equiv |\uparrow\downarrow\uparrow\cdots\downarrow\rangle, \quad |\mathcal{N}_2\rangle \equiv |\downarrow\uparrow\downarrow\cdots\uparrow\rangle$$

Intuitive requirements for true ground-state $|A\rangle$:

$\rightarrow$ full rotational invariance

$\rightarrow$ zero magnetization, i.e. $r = N/2$

Starting from ferromagnetic case:

Construction via excitation of $N/2$ (interacting) magnons from $|F\rangle$

$$|A\rangle = \sum_{1\leq n_1<\ldots<n_r\leq N} a(n_1,\ldots,n_r)|n_1,\ldots,n_r\rangle \quad \text{with} \quad r = N/2$$

Ground-state

finite N study reveals

$$|A\rangle \quad \Leftrightarrow \quad \{\lambda_i\}_A = \{1, 3, 5, \ldots, N-1\}$$

Ground-state

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$$|A\rangle \quad \Leftrightarrow \quad \{\lambda_i\}_A = \{1, 3, 5, \ldots, N-1\}$$

quantum numbers {λ_i}	**quantum numbers {I_i}**
parametrization $\{k_i\}$, $\{\theta_{ij}\}$	parametrization $\{z_i\}$ obtained as
	$k_i \equiv \pi - \phi(z_i)$ where $\phi(z) \equiv 2 \arctan z$
$2 \cot \frac{\theta_{ij}}{2} = \cot \frac{k_i}{2} - \cot \frac{k_j}{2}$	$\theta_{ij} = \pi \, \mathrm{sgn}[\Re(z_i - z_j)] - \phi\big[(z_i - z_j)/2\big]$
$Nk_i = 2\pi\lambda_i + \sum_{j\neq i} \theta_{ij}$	$N\phi(z_i) = 2\pi I_i + \sum_{j\neq i} \phi\big[(z_i - z_j)/2\big]$

such that

$$|A\rangle \quad \Leftrightarrow \quad \{I_i\}_A = \left\{-\frac{N}{4} + \frac{1}{2}, \ -\frac{N}{4} + \frac{3}{2}, \ldots, \frac{N}{4} - \frac{1}{2}\right\}$$

Ground-state

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obtain z_i and with that wave numbers k_i by fixed point iteration

$$N\phi(z_i) = 2\pi I_i + \sum_{j\neq i}\phi\big[(z_i - z_j)/2\big]$$

$$\Rightarrow z_i^{(n+1)} = \tan\left(\tfrac{\pi}{N} I_i + \tfrac{1}{2N}\sum_{j\neq i} 2\arctan\big[(z_i^{(n)} - z_j^{(n)})/2\big]\right)$$

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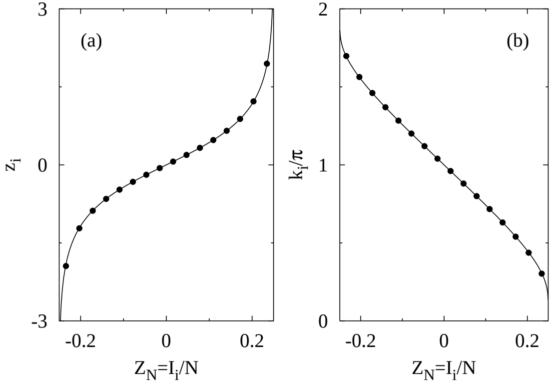

Karbach, Hu, and Müller, Computers in Physics (1998)

Energy in the thermodynamic limit

$(E - E_F)/J = \sum_{i=1}^{r} \varepsilon(z_i)$ where $\varepsilon(z_i) = -2/(1 + z_i^2)$

(remember $(E - E_F)/J = \sum_{i=1}^{r}(1 - \cos k_i)$)

where the sum is over $I_i \in \left\{ -\frac{N}{4} + \frac{1}{2},\ -\frac{N}{4} + \frac{3}{2}, \ldots, \frac{N}{4} - \frac{1}{2} \right\}$

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For $N \to \infty$ define continuous variable $I \equiv I_i/N$

$$(E - E_F)/(JN) = \frac{1}{N} \sum_{i=1}^{r} \varepsilon(z_i)$$

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where

$$\sigma_0 \equiv \frac{\mathrm{d}I}{\mathrm{d}z} = \frac{1}{4\cosh(\pi z/4)} \quad \text{from} \quad N\phi(z_i) = 2\pi I_i + \sum_{j \neq i} 2 \arctan\left[(z_i - z_j)/2\right]$$

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where the sum is over $I_i \in \left\{-\frac{N}{4} + \frac{1}{2},\ -\frac{N}{4} + \frac{3}{2}, \ldots, \frac{N}{4} - \frac{1}{2}\right\}$

For $N \to \infty$ define continuous variable $I \equiv I_i/N$

$$(E - E_F)/(JN) = \frac{1}{N}\sum_{i=1}^{r}\varepsilon(z_i) = \frac{1}{N}\sum_{I_i=-\frac{N}{4}+\frac{1}{2}}^{\frac{N}{4}-\frac{1}{2}}\varepsilon(z_{I_i}) = \int_{-1/4}^{1/4}\mathrm{d}I\,\varepsilon(z_I) = \int_{-\infty}^{\infty}\mathrm{d}z\,\sigma_0\varepsilon(z_I)$$

where

$$\sigma_0 \equiv \frac{\mathrm{d}I}{\mathrm{d}z} = \frac{1}{4\cosh(\pi z/4)} \quad \text{from} \quad N\phi(z_i)=2\pi I_i+\sum_{j\neq i} 2\arctan\left[(z_i-z_j)/2\right]$$

such that energy

$$(E - E_F)/(JN) = \ln 2$$

Magnetic field

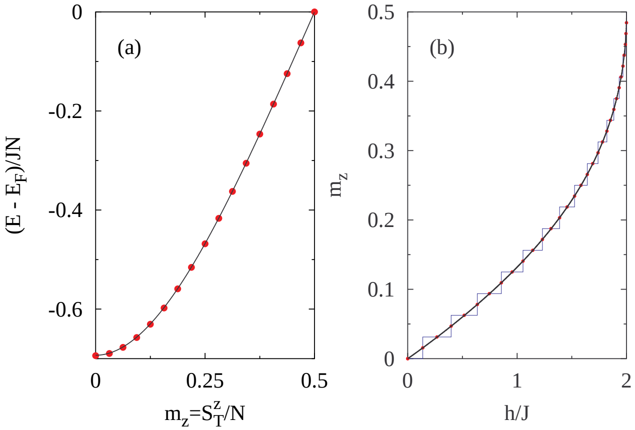
$$H = J\sum_{n=1}^{N} \mathbf{S}_n \cdot \mathbf{S}_{n+1} - h\sum_{n=1}^{N} S_n^z$$

If field *h* strong enough

$|F\rangle \equiv |\uparrow \ldots \uparrow\rangle$ will become ground-state

- groundstate $|A\rangle$ for very small h
- $|F\rangle$ "overtakes" all other states with increasing h
- saturation field $h_S = 2J$ (=energy difference between state $|F\rangle$ and $|k = 0\rangle$)

Magnetization



Karbach, Hu, and Müller, Computers in Physics (1998)

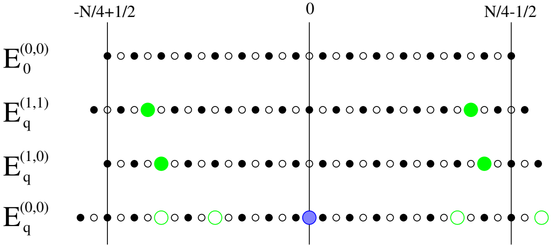
susceptibility

infinite slope at the saturation field is pure quantum feature

Two-spinon excitations

ground-state

$$|A\rangle \quad \Leftrightarrow \quad \{I_i\}_A = \left\{-\frac{N}{4}+\frac{1}{2},\ -\frac{N}{4}+\frac{3}{2},\ldots,\frac{N}{4}-\frac{1}{2}\right\}$$



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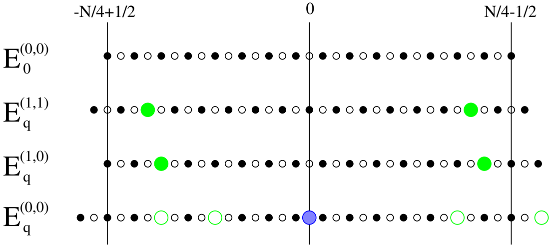

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Fundamental excitations are pairs of spinons

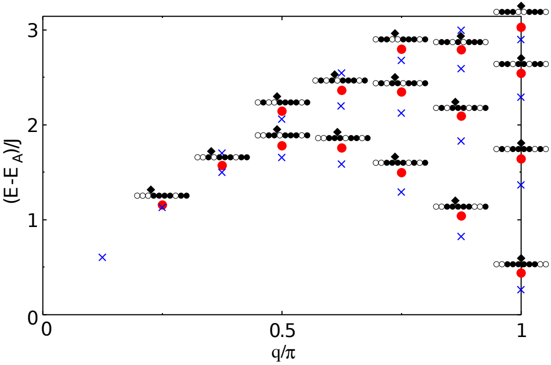
- magnon picture: remove one magnon from $|A\rangle$ $(N/2 \to N/2 - 1$ quantum numbers)
- spinon picture: representation as array (gaps are spinons)

Note: Spinons spin-1/2 particles, Magnons spin-1 particles

Two-spinon excitations: dispersion

Sum of two spinon wave numbers $q = \bar{k}_1 + \bar{k}_2$

in contrast to $N/2 - 1$ wave numbers k_i in magnon picture



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dispersion boundaries : $\epsilon_L(q) = \frac{\pi}{2} J |\sin q|, \quad \epsilon_U(q) = \pi J |\sin \frac{q}{2}|$

Contents

Examples for models

- Heisenberg model

$$H = \pm J \sum_i \left[\frac{1}{2} \left(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+ \right) + S_i^z S_{i+1}^z \right]$$

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$$\xrightarrow{\text{s-wave + low energy}} H = -i \int \mathrm{d}x \; \psi(x)_s^\dagger \partial_x \psi(x)_s + \psi(x = 0)_s^\dagger \boldsymbol{\sigma}_{ss'} \psi(x = 0)_{s'} \cdot \boldsymbol{\sigma}_0$$

Hubbard model

First steps of systematic solution allow to elucidate fundamental principles.

Hilbert space of N particles spanned by

$$|\psi\rangle = \sum_{n_1,\ldots,n_N} a_{s_1,\ldots,s_N}(n_1,\ldots,n_N) \prod_i c^\dagger_{n_i s_i} |\text{vac}\rangle$$

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Thus

$$H|\psi\rangle = E|\psi\rangle \quad \longrightarrow \quad h\boldsymbol{a} = E\boldsymbol{a}$$

with

$$h = -t \sum_j \Delta_j + U \sum_{j<l} \delta_{n_j n_l}$$

Hubbard model

Take large lattice $L \to \infty$

One particle case

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solved by plane waves

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Consider $n_1 = n_2 = n$ as third boundary for the system

- System consists of two regions $A \cap B \equiv [-L, n] \cap [n, L]$.
- Clearly, in both regions the Hamiltonian is of non-interacting form!
- In these subsets the solutions are given by plane waves again!

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ansatz:

$$a_{s_1,s_2}(n_1, n_2) = \mathcal{A}e^{ik_1 n_1 + ik_2 n_2}(\underbrace{A_{s_1,s_2}\Theta(n_1 - n_2)}_{\text{wavefunction in subset A}} + \underbrace{B_{s_1,s_2}\Theta(n_2 - n_1)}_{\text{wavefunction in subset B}})$$

Note: remember the Heisenberg model

- block $r = 1$:

$$|\psi\rangle = \sum_{n=1}^{N} a(n)|n\rangle$$

$H|\psi\rangle = E|\psi\rangle \Leftrightarrow$

$$2[E - E_0]a(n) = J\underbrace{[2a(n) - a(n-1) - a(n+1)]}_{= \Delta a(n)}$$

- block $r = 2$:

$$|\psi\rangle = \sum_{1 \le n_1 < n_2 \le N} a(n_1, n_2)|n_1, n_2\rangle \quad \text{where} \quad |n_1, n_2\rangle \equiv S^-_{n_1} S^-_{n_2}|F\rangle$$

$H|\psi\rangle = E|\psi\rangle \longrightarrow$

$$\text{for} \quad n_2 > n_1 + 1 : \quad 2[E - E_0]a(n_1, n_2) =$$

$$= J\underbrace{[4a(n_1, n_2) - a(n_1 - 1, n_2) - a(n_1 + 1, n_2) - a(n_1, n_2 - 1) - a(n_1, n_2 + 1)]}_{= (\Delta_1 + \Delta_2)a(n_1, n_2)}$$

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S-matrix and generalization to N particles (back to Hubbard model)

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- Describes scattering processes in the basis of free particles!
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Summarize this viewpoint

- Hubbard, Heisenberg and Kondo model subject to local interaction.
- In the “free” regions, plain waves constitute solutions.
- Amplitudes of “free” regions related by two-particle S-matrix.

Generalization to N particles, Yang Baxter Equation

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$$a_{s_1,\dots s_N} = \mathcal{A} e^{\sum_j k_j n_j} \sum_{\mathcal{P}_R} A_{s_1,\dots,s_N}(\mathcal{P}_R)\Theta(n_{\mathcal{P}_R})$$

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$N = 3$ particles:
$\rightarrow$ relate the amplitudes of two different regions R_1 and R_2

$$A(\mathcal{P}_{R_1}) = S^{ij} S^{jk} S^{kl} A(\mathcal{P}_{R_2})$$

Usually there are several ways to relate different regions. The consistency of the ansatz requires uniqueness for different paths, i.e.

$$S^{23} S^{13} S^{12} = S^{12} S^{13} S^{23}$$

$\rightarrow$ **Yang-Baxter equation** for three particles

Contents

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- But, no problem: all this guaranteed by successful check of the Yang-Baxter equation.

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Thank you for your attention!