Statistical description of prethermalization plateaus

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Quench of an isolated quantum system

- Sudden change of a parameter

$$H = \begin{cases} H_0 & \text{for } t < 0 \\ H_0 + g\, H_1 & \text{for } t \geq 0 \end{cases}$$

- Time evolution of the state

$$|\Psi(t<0)\rangle \equiv |\Psi_0\rangle$$
$$|\Psi(t \geq 0)\rangle = \exp(-iHt)|\Psi_0\rangle$$

and of the expectation value of an observable A

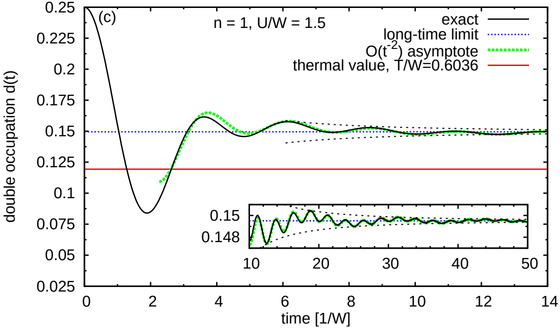
$$\langle\Psi(t)|A|\Psi(t)\rangle$$

Integrable systems

- No thermalization!

M. A. Cazalilla, PRL **97** (2006)
M. Rigol et al, PRL **98** (2006)
M. Eckstein and M. Kollar, PRL **100** (2008)
M. Kollar and M. Eckstein, PRA **78** (2008) $\longrightarrow$ Figure

$$\langle\Psi(t)|A|\Psi(t)\rangle_{t\to\infty} \neq \langle A\rangle_{\text{therm}}$$



1/r-Hubbard model in $d = 1$, $U = 0 \rightarrow U = 1.5$

Why is there no thermalization?

- Conservation of many ($\propto L$) constants of motion I_α
- Much fewer accessible states in relaxation process
- Failure of the assumption of standard statistical mechanics:

  "All states with the same energy are equally probable."
- $\rho_{\mathrm{Gibbs}} \equiv e^{-\beta H}$ not appropriate

Statistical description of non-thermal longtime limit

- "Generalized" Gibbs Ensemble ρ_{GGE} often correct

M. Rigol et al, PRL **98** (2007)
M. Kollar and M. Eckstein, PRA **78** (2008)

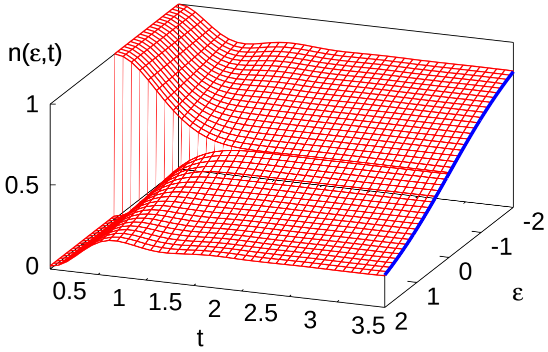
$$\rho_{\mathrm{GGE}} := e^{-\sum_\alpha \lambda_\alpha I_\alpha},$$

where $\langle I_\alpha \rangle_{\mathrm{GGE}} \stackrel{!}{=} \langle I_\alpha \rangle_0$ fixes λ_α

Non-integrable systems

- Thermalization possible!

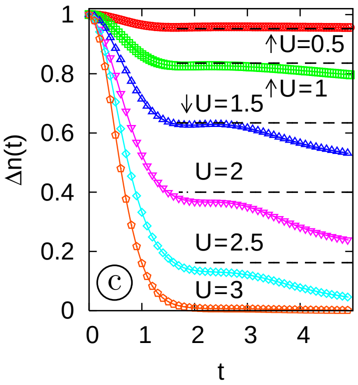
M. Rigol et al, Nature **452** (2008)
P. Barmettler et al, PRL **102** (2009)
M. Eckstein, M. Kollar, P. Werner, PRL **103** (2009)→ Figure



Hubbard model in $d = \infty$, $U = 0 \rightarrow U = 3$

Non-int. systems close to an integrable point

- Intermediate time scales: prethermalization = meta-stable state before later thermalization
- Long time scales: thermalization



J. Berges et al, PRL **93** (2004)
M. Moeckel and S. Kehrein, PRL **100** (2008)
M. Eckstein, M. Kollar, P. Werner, PRL **103** (2009) → Figure

Hubbard model in $d = \infty$

Today

Question:
Statistical description of prethermalization plateaus possible?

Answer:
Yes, with appropriate GGE under certain conditions.

Outline

Assumptions

Before quench:

- $H_0 = \sum_\alpha \epsilon_\alpha I_\alpha,\ I_\alpha = a^\dagger_\alpha a_\alpha \ \Rightarrow\ [I_\alpha, I_\beta] = 0,\ [H_0, I_\alpha] = 0$
- System in ground state $|\Psi_0\rangle$ of H_0
- Define basis $\{|\boldsymbol{n}\rangle\}$, $|\boldsymbol{0}\rangle \equiv |\Psi_0\rangle$ with

$$I_\alpha|\boldsymbol{n}\rangle = n_\alpha|\boldsymbol{n}\rangle$$
$$H_0|\boldsymbol{n}\rangle = E_{\boldsymbol{n}}|\boldsymbol{n}\rangle$$

After quench:

- $H = H_0 + g\,H_1,\ g \ll 1\,,\quad g =$ strength of interaction
- $[H_0, H_1] \neq 0$
- H_1 can be expressed with $a^\dagger_\alpha$ and a_α

Example:

- Quench of the non-interacting Hubbard model to small U

Construction of approx constants of motion

Need: Operator-based perturbation theory

A. Harris and V. Lange, PR **157** (1963)
M. Eckstein, Dissertation (2009)

Canonical trafo of H with generator $S = gS_1 + \frac{g^2}{2}S_2$

$$\begin{aligned} e^S H e^{-S} =& H_0 + g(H_1 + [S_1, H_0]) \\ & + g^2(\tfrac{1}{2}[S_2, H_0] + [S_1, H_1] + \tfrac{1}{2}[S_1, [S_1, H_0]]) + O(g^3) \end{aligned}$$

Demand $[e^S H e^{-S}, I_\alpha] = 0$ order by order

- Use eigenbasis $\{|\boldsymbol{n}\rangle\}$ of I_α and H_0
- If H_0 degenerate, choose basis such that

$$\langle \boldsymbol{n}|H_1|\boldsymbol{m}\rangle \propto \delta_{\boldsymbol{nm}} \quad \text{if} \quad E_{\boldsymbol{n}} = E_{\boldsymbol{m}}$$

- Result: only off-diagonal elements of S nonzero

$$\text{For } E_{\boldsymbol{n}} \neq E_{\boldsymbol{m}} : \quad (S_1)_{\boldsymbol{nm}} = \frac{1}{E_{\boldsymbol{n}} - E_{\boldsymbol{m}}} (H_1)_{\boldsymbol{nm}}$$

$$(S_2)_{\boldsymbol{nm}} = \frac{1}{E_{\boldsymbol{n}} - E_{\boldsymbol{m}}} ([S_1, H_1 + \text{diag}(H_1)])_{\boldsymbol{nm}}$$

- Transformed Hamiltonian

$$\Longrightarrow \quad e^S H e^{-S} = H_0 + \underbrace{g\, h_{\text{diag}1} + g^2 h_{\text{diag}2}}_{=: H_{\text{diag}}} + O(g^3)$$

$$\text{where} \quad h_{\text{diag}1} = \sum_{\boldsymbol{n}} |\boldsymbol{n}\rangle \underbrace{\langle \boldsymbol{n}| H_1 |\boldsymbol{n}\rangle}_{= E_{\boldsymbol{n}}^{(1)}} \langle \boldsymbol{n}|$$

$$h_{\text{diag}2} = \sum_{\boldsymbol{n}} |\boldsymbol{n}\rangle \underbrace{\sum_{m \neq n} \frac{\langle \boldsymbol{m}| H_1 |\boldsymbol{n}\rangle}{E_{\boldsymbol{n}} - E_{\boldsymbol{m}}}}_{= E_{\boldsymbol{n}}^{(2)}} \langle \boldsymbol{n}|$$

- Inverse transformation:

$$H = e^{-S} H_0 e^{S} + e^{-S} H_{\text{diag}} e^{S} + O(g^3)$$

- Define: $\widetilde{I}_\alpha := e^{-S} I_\alpha e^{S}$ and $|\widetilde{\boldsymbol{n}}\rangle := e^{-S} |\boldsymbol{n}\rangle$

$$H = \sum_\alpha \epsilon_\alpha \, \widetilde{I}_\alpha + \sum_{\widetilde{\boldsymbol{n}}} |\widetilde{\boldsymbol{n}}\rangle\langle\widetilde{\boldsymbol{n}}| \; (g E^{(1)}_{\boldsymbol{n}} + g^2 E^{(2)}_{\boldsymbol{n}}) + O(g^3)$$

- $|\widetilde{\boldsymbol{n}}\rangle$ eigenstate of $\widetilde{I}_\alpha$ and H in order $O(g^2)$
- $\widetilde{I}_\alpha$ approx constant of motion in the quenched system!

$$[H, \widetilde{I}_\alpha] = 0, \quad [\widetilde{I}_\alpha, \widetilde{I}_\beta] = 0$$

Construction of the GGE

- Use approximate constants of motion $\widetilde{I}_\alpha$ to generate a GGE:

$$\rho_{\widetilde{\mathrm{GGE}}} := e^{-\sum_\alpha \lambda_\alpha \widetilde{I}_\alpha}$$

- $\rho_{\widetilde{\mathrm{GGE}}}$ maximizes the entropy under the constraints that

$$\langle \widetilde{I}_\alpha \rangle_{\widetilde{\mathrm{GGE}}} \stackrel{!}{=} \langle \widetilde{I}_\alpha \rangle_0 \equiv \langle \Psi_0 | \widetilde{I}_\alpha | \Psi_0 \rangle$$

Evaluation of the GGE

- Evaluation for an explicit choice of the observable A

$$A := \prod_{i=1\ldots m} I_{\alpha_i} \quad \Rightarrow \quad [A, H_0] = 0$$

- Apply transformation to $\langle A \rangle_{\widetilde{\mathrm{GGE}}}$

$$\begin{aligned}\langle A \rangle_{\widetilde{\mathrm{GGE}}} &= \frac{1}{Z}\mathrm{Tr}[A\, e^{-\sum_\alpha \lambda_\alpha \widetilde{I}_\alpha}] \\ &= \frac{1}{Z}\mathrm{Tr}[e^S A e^{-S}\, e^{-\sum_\alpha \lambda_\alpha I_\alpha}] = \langle e^S A e^{-S} \rangle_{\mathrm{GGE}}\end{aligned}$$

- Evaluate transformed GGE

$$\langle e^S A e^{-S} \rangle_{\text{GGE}} = \underbrace{\langle A \rangle_{\text{GGE}}}_{=(\text{i})} + \underbrace{\langle [S, A] \rangle_{\text{GGE}}}_{=0} + \underbrace{\langle \tfrac{1}{2}[S, [S, A]] \rangle_{\text{GGE}}}_{=(\text{ii})} + O(g^3)$$

$$\begin{aligned}
(\text{i}) &= \langle A \rangle_{\text{GGE}} \\
&= \Big\langle \prod_{i=1\ldots m} I_{\alpha_i} \Big\rangle_{\text{GGE}} \\
&= \prod_{i=1\ldots m} \langle I_{\alpha_i} \rangle_{\text{GGE}} \quad \text{common eigenbasis} \\
&= \prod_{i=1\ldots m} \langle \Psi_0 | \widetilde{I}_{\alpha_i} | \Psi_0 \rangle \quad \text{fix initial value} \\
&= \prod_{i=1\ldots m} \langle \widetilde{\Psi}_0 | I_{\alpha_i} | \widetilde{\Psi}_0 \rangle + O(g^3) \quad \text{state transformation}
\end{aligned}$$

$$
\begin{aligned}
\text{(ii)} =&\langle \tfrac{1}{2}[S,[S,A]]\rangle_{\text{GGE}} \\
&= \frac{g^2}{Z}\sum_{\boldsymbol{n}} \underbrace{\langle \boldsymbol{n}|\tfrac{1}{2}[S_1,[S_1,A]]|\boldsymbol{n}\rangle}_{=:\,F(\{n_\alpha\})} e^{-\sum_\alpha \lambda_\alpha n_\alpha} + O(g^3) \\
&= g^2 F\big(\{\langle I_\alpha\rangle_{\text{GGE}}\}\big) + O(g^3) \quad \text{Wick's theorem} \\
&= g^2 F\big(\{\langle \Psi_0|\widetilde{I}_\alpha|\Psi_0\rangle\}\big) + O(g^3) \quad \text{fix initial value} \\
&= g^2 F\big(\{\langle \Psi_0|I_\alpha|\Psi_0\rangle\}\big) + O(g^3) \quad \text{lowest order} \\
&= g^2 \langle \Psi_0|\tfrac{1}{2}[S_1,[S_1,A]]|\Psi_0\rangle + O(g^3) \quad \text{same lin. comb.} \\
&= \langle \Psi_0|\tfrac{1}{2}[S,[S,A]]|\Psi_0\rangle + O(g^3)
\end{aligned}
$$

$$
\begin{aligned}
(\mathrm{ii}) &= \langle \tfrac{1}{2}[S,[S,A]]\rangle_0 \\
&= \langle (A + \tfrac{1}{2}[S,[S,A]])\rangle_0 - \langle A\rangle_0 + O(g^3) \\
&= \langle \Psi_0|\widetilde{A}|\Psi_0\rangle - \langle A\rangle_0 + O(g^3) \\
&= \langle \widetilde{\Psi_0}|A|\widetilde{\Psi_0}\rangle - \langle A\rangle_0 + O(g^3) \\
&= \langle \widetilde{\Psi_0}| \prod_{i=1\dots m} I_{\alpha_i}|\widetilde{\Psi_0}\rangle - \prod_{i=1\dots m} \langle I_{\alpha_i}\rangle_0 + O(g^3)
\end{aligned}
$$

- Final result for GGE expectation value: (i) + (ii)

$$
\langle A\rangle_{\widetilde{\mathrm{GGE}}} = \langle \prod_{i=1\dots m} I_{\alpha_i}\rangle_{\widetilde{\mathrm{GGE}}}
$$

$$
= \prod_{i=1\dots m} \langle \widetilde{\Psi_0}|I_{\alpha_i}|\widetilde{\Psi_0}\rangle + \langle \widetilde{\Psi_0}| \prod_{i=1\dots m} I_{\alpha_i}|\widetilde{\Psi_0}\rangle - \prod_{i=1\dots m} \langle I_{\alpha_i}\rangle_0 + O(g^3)
$$

Comparison to explicit time evolution

- Prethermalization plateau M. Moeckel and S. Kehrein, Ann. Phys. **324** (2009)

$$\overline{\langle A\rangle} = \lim_{t\to\infty}\frac{1}{t}\int_0^t \mathrm{d}t\langle\Psi_0|e^{\mathrm{i}Ht}Ae^{-\mathrm{i}Ht}|\Psi_0\rangle$$
$$= 2\,\langle\widetilde{\Psi_0}|\prod_{i=1\ldots m} I_{\alpha_i}|\widetilde{\Psi_0}\rangle - \prod_{i=1\ldots m}\langle I_{\alpha_i}\rangle_0 + O(g^3)$$

for timescales $\frac{1}{g} \ll t \ll \frac{1}{g^2}$

- Condition: $\langle A\rangle_{\widetilde{\mathrm{GGE}}} = \overline{\langle A\rangle} + O(g^3)$ holds if

$$\prod_{i=1\ldots m}\langle\widetilde{\Psi_0}|I_{\alpha_i}|\widetilde{\Psi_0}\rangle \stackrel{!}{=} \langle\widetilde{\Psi_0}|\prod_{i=1\ldots m} I_{\alpha_i}|\widetilde{\Psi_0}\rangle + O(g^3)$$

Remarks

- Condition trivially fulfilled for $m = 1 \rightarrow A = I_\alpha \equiv N_\alpha$

$$\langle N_\alpha \rangle_{\widetilde{\text{GGE}}} = \overline{N_\alpha} + O(g^3)$$

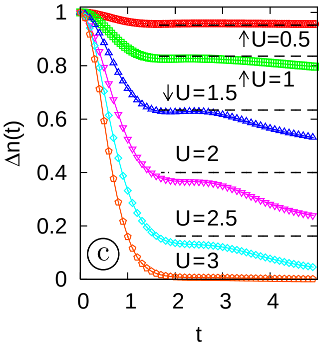
- Condition analogous to that for integrable systems

  → M. Kollar and M. Eckstein, PRA **78** (2008)

- Meaning of condition: Observable and/or prethermalized state must not be too correlated (similar to standard statistical mechanics)

Summary

- First statistical description of a prethermalization plateau
- Statistical mechanics works! (if done correctly)



Thanks to Marcus Kollar and Dieter Vollhardt

Thank your for your attention!